Attacks on Schnorr signatures with biased nonces

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Outline

Schnorr signatures with biased nonces
- Schnorr signatures
- Nonce biases

The lattice approach
- Description of the attack
- Limitations and extensions

The statistical approach
- Attack overview
- Using Schroeppel–Shamir

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Schnorr signatures

- Public parameters: cyclic group $\mathbb{G}$ of prime order q, generator g, hash function $H: \{0,1\}^* \times \mathbb{G} \to \mathbb{Z}/q\mathbb{Z}$
- Key pair: secret $x \xleftarrow{\$} \mathbb{Z}/q\mathbb{Z}$, public $h = g^x$

$\mathrm{Sign}(x, m)$

1. $k \xleftarrow{\$} \mathbb{Z}/q\mathbb{Z}$
2. $r \leftarrow g^k$
3. $h \leftarrow H(m, r)$
4. $s \leftarrow k - hx \bmod q$
5. **return** (h, s)

EC-Schnorr signatures

- Public parameters: elliptic curve $E/\mathbb{F}_p$, point $P \in E(\mathbb{F}_p)$ of prime order q, hash function $H\colon \{0,1\}^* \times \mathbb{F}_p \to \mathbb{Z}/q\mathbb{Z}$
- Key pair: secret $x \xleftarrow{\$} \mathbb{Z}/q\mathbb{Z}$, public $Q = [x]P$

$\mathrm{Sign}(x, m)$

1. $k \xleftarrow{\$} \mathbb{Z}/q\mathbb{Z}$
2. $(u, v) \leftarrow [k]P$
3. $h \leftarrow H(m, u)$
4. $s \leftarrow k - hx \bmod q$
5. **return** (h, s)

Security and variants

- Secure (EUF-CMA) if the discrete logarithm is hard in $\mathbb{G}$ in the ROM for H
- Common variants:
  - Hash the public key as well
  - Deterministic k, e.g. $k = \mathrm{MAC}_S(m)$ for an auxiliary key S
  - Give out g^k (resp. $k[P]$) instead of h in the signature
- EdDSA ≈ EC-Schnorr on X25519; qDSA ≈ Schnorr on Kummer lines/Kummer surfaces
- DSA, ECDSA: badly designed variants of Schnorr (for patent reasons)
- Results in this talk apply to all of the above
  - Caveat: specific ways of inducing nonce biases may only apply to a subset of them

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Sensitivity of the nonce

- The random value k in signature generation usually called the nonce
- Should never be repeated! If (h, s), (h', s') signatures on m, m' with the same value k, we have:

  $$s \equiv k - hx \pmod{q} \qquad s' \equiv k - h'x \pmod{q}$$

  Subtract the two:

  $$(h' - h) \cdot x \equiv s - s' \pmod{q}$$

  Immediate recovery of the secret key x
- That attack (for ECDSA) was applied to the Sony PlayStation 3. Also used to steal some Bitcoins

Sensitivity of the nonce (cont'd)

- Nonce = value that can only be used once
- However Schnorr nonces are even more sensitive than that!
- k should (statistically close to) uniform in $\mathbb{Z}/q\mathbb{Z}$. Significant biases can be used reveal the key
- Intuition: linear relation

$$x = h^{-1} \cdot (-s + k) \bmod q$$

$\rightsquigarrow$ partial info. on k (e.g. ℓ known bits) should translate to partial info. on x (ℓ bits of info)

How do biases occur?

- Incorrect implementation
  - PlayStation 3
  - GLV/GLS setting: k implicitly defined as $k_1 + \lambda k_2$ with k_1, k_2 of roughly half size; if "half size" is interpreted as $\lceil (\log_2 q)/2 \rceil$ bits, bias can occur
- Poor random number generators
- Side-channel leakage
  - e.g. emanations during scalar multiplication revealing the first few LSBs of k
- Fault attacks
  - errors injected before/during the computation of $[k]P$ forcing k to a biased value

Classical fault attack on ECDSA

- Successfully demonstrated by Naccache et al. against 8-bit smartcards (PKC 2005)
- Upon signature generation, new k generated uniformly at random in $\mathbb{Z}/q\mathbb{Z}$
- Typically done machine word by machine word (sample each word with a random number generator), with rejection sampling at the end
- Glitch attack: inject a fault during the random sampling loop to cause an early exit, so LSBs or MSBs of k are left equal to zero
- Usual countermeasure: use double loop counters

New: a fault attack on qDSA

- Fault injection on a much less protected part of signature generation: the group generator P
  - usual ECDSA/EC-Schnorr: a random fault yields a point $\widetilde{P}$ outside the curve $\rightsquigarrow$ scalar multiplication makes no sense
  - qDSA: x-only arithmetic on X25519 $\rightsquigarrow$ result on the curve or its twist
- With prob. $\approx 1/4$, the faulty generator $\widetilde{P}$ is on the curve itself and has order $4q$
- qDSA signatures include $\pm R = \pm[k]P$ instead of the hash
  - faulty case $[q]\widetilde{R} = [k]\big([q]\widetilde{P}\big)$ point of order 4, revealing the 2 LSBs of k
  - only known up to sign $\rightsquigarrow$ deduce if the 2 LSBs are 00, 10 or ?1
- Suppose we can generate many signatures with the same $\widetilde{P}$ (semi-permanent fault setting)
  - Can check that $\widetilde{P}$ has order $4q$
  - Mount attack with 2-bit nonce bias (throw away the ?1 case)

Exploiting the nonce biases

- Given biases nonces k, two main approaches to recover x
- Lattice-based attack (Howgrave-Graham–Smart; Nguyen–Shparlinski)
  - based on solving BDD in a lattice
  - requires relatively few signatures
  - for large biases ($\gtrsim 5$ bits depending on the size of q), very efficient in practice
  - for small biases, impractical (lattice dim. too large) or even inapplicable (hidden vector not close enough)
  - cannot use more data
  - bias must be "predictable"
- Statistical attack (Bleichenbacher)
  - based on purely statistical techniques (FFT)
  - requires many signatures, large space complexity
  - can in principle deal with arbitrarily small biases
  - more data improves the attack
  - irregular biases OK

Current records

- Lattice-based attack
  - 160 bits: 2-bit bias done ([LN13], ≈ 100 sigs., BKZ–90), 1-bit infeasible
  - 256 bits: 4-bit easy, 3-bit not easy, 2-bit infeasible?
  - 384 bits: 6-bit easy, 5 or 4-bit not so easy, 3-bit infeasible?
- Statistical attack
  - 160 bits: 1-bit bias done ([AFGKTZ14], $\approx 2^{30}$ sigs., 1 TB RAM)
  - 256 bits: 2-bit looks hard, 1-bit possible with nation-state resources and many sigs.?
  - 384 bits: 5-bit done ([DHMP13], ≈ 4000 sigs.), 4-bit feasible?, 3-bit hard?

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Formal attack setting

- We obtain n faulty signatures (h_i, s_i) on messages m_i
- Each signature generated with nonce k_i with ℓ LSBs equal to zero:

$$k_i = 2^\ell b_i, \quad (0 \le b < q/2^\ell)$$

- We thus get relations:

$$h_i \cdot x \equiv 2^\ell b_i - s_i \pmod{q}$$

which we can rewrite as:

$$x \equiv u_i + v_i b_i \pmod{q}$$

for known constants $u_i = -s_i/h_i \bmod q$, $v_i = 2^\ell/h_i \bmod q$.

Orthogonal lattices

- Rewrite the previous relation in vector form:

$$x \equiv \langle \mathbf{u}_i, \mathbf{b} \rangle \pmod{q}$$

with:

$$\mathbf{b} = (a, b_1, \dots, b_n) \in \mathbb{Z}^{n+1}$$
$$\mathbf{u}_i = (u_i/a \bmod q, 0, \dots, 0, v_i, 0, \dots, 0) \in \mathbb{Z}^{n+1}$$

- In particular, $\mathbf{b}$ orthogonal mod q to $\mathbf{u}_1 - \mathbf{u}_2$, $\mathbf{u}_2 - \mathbf{u}_3$, ..., $\mathbf{u}_{n-1} - \mathbf{u}_n$
- Introduce the lattice L of vectors in $\mathbb{Z}^{n+1}$ orthogonal to those $n-1$ vectors mod q, and such that the first component is a multiple of a
- $\mathbf{b} \in L$, relatively short

Recovering b

- L is the kernel of the map $\mathbb{Z}^{n+1} \to (\mathbb{Z}/a\mathbb{Z}) \times (\mathbb{Z}/q\mathbb{Z})^{n-1}$:

$$\mathbf{b} \mapsto \big(b_1 \bmod a, \langle \mathbf{b}, \mathbf{u}_1 - \mathbf{u}_2 \rangle \bmod q, \dots, \langle \mathbf{b}, \mathbf{u}_{n-1} - \mathbf{u}_n \rangle \bmod q\big)$$

  surjective with high probability
- Therefore $\mathrm{vol}(L) = [\mathbb{Z}^{n+1} : L] = a \cdot q^{n-1}$
- Gaussian heuristic: the shortest vector in L should be of length approximately

$$\lambda = \frac{n+1}{2\pi e} \cdot \mathrm{vol}(L)^{1/(n+1)}$$

- We can hope to recover $\mathbf{b}$ if $\|\mathbf{b}\| \ll \lambda$. Choosing $a = q/2^\ell$, we have $\mathbf{b} \leq \sqrt{n+1} \cdot q/2^\ell$
- Recovering $\mathbf{b}$ of course reveals the secret key x

Condition on n

- The size condition is thus:

$$\sqrt{n+1}\cdot\frac{q}{2^\ell} \ll \sqrt{\frac{n+1}{2\pi e}}\left(\frac{q}{2^\ell}\cdot q^{n-1}\right)^{1/(n+1)}$$

  which simplifies to:

$$\frac{n}{n+1}\ell \gtrsim \frac{1}{n+1}\log_2 q + \log_2\sqrt{2\pi e}$$

- In particular, the attack only works when $\ell > \log_2\sqrt{2\pi e} \approx 2.05)$
  - constant slightly too large: can be improved by using a centered **b**, and taking expected size into account
- For fixed ℓ, we need a number of faulty signatures satisfying:

$$n \gtrsim \frac{\log_2(q\sqrt{2\pi e})}{\ell - \log_2\sqrt{2\pi e}}$$

- Large ℓ: close to "information-theoretic" bound $(\log_2 q)/\ell$

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Remarks on the attack

- Different from traditional presentation of the attack (uSVP vs. BDD), but mostly equivalent
- Must know the size of the bias(es) to construct the lattice
- As already mentioned: hard limit on how small the bias can get
- Having more signatures doesn't seem to help

Structure of the bias

- Attack uses in a crucial way the zero LSBs form of the bias
  - known LSBs/MSBs of course also OK
- Does not generalize easily to more general bias structure
  - doable: string of zero bits in the middle at known position
  - hard?: string of zero bits in the middle at unknown position
- Recent generalizations with some practical relevance
  - zero LSBs/MSBs in the τ-adic expansion of k for Kobliz curves [BFMT16]
  - zero LSBs/MSBs in k_i for $k = \sum k_i \lambda_i$ GLV/GLS decomposition
  - fun fact: can use lattice reduction over Euclidean rings
- Hard to formulate general result?

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Bleichenbacher's attack

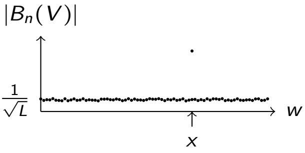
- Before the lattice attack was proposed, Bleichenbacher suggested a different approach based on a Fourier notion of bias
- Requires many more signatures for similar parameters, but applies in principle to arbitrarily small biases
- Presented at an IEEE P1363 meeting in 2000, never formally published. Revisited by De Mulder et al. (CHES 2013), Aranha et al. (ASIACRYPT 2014).

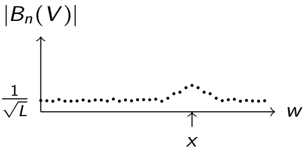
Overview of Bleichenbacher's approach (I)

- Consider again: we are given signatures (h_j, s_j) such that, for the secret key x, the MSBs of the values $k_j = s_j + h_j x \bmod q$ vanish.
- The sampled bias of a set of points $V = (v_0, \cdots, v_{L-1})$ in $\mathbb{Z}/q\mathbb{Z}$ defined as $B_q(V) = \frac{1}{L}\sum_{j=0}^{L-1} e^{2\pi i \cdot v_j/q}$
- Now consider some secret key candidate $w \in \mathbb{Z}/q\mathbb{Z}$ and the corresponding nonce candidates $v_j := s_j + h_j w \mod q$. Claim:
  - if $w \neq x$, $B_q(V) \approx 1/\sqrt{L}$ is small
  - if $w = x$, $B_q(V)$ is close to 1

Range reduction

- Peak of the bias function: only for candidate w exactly equal to x; would need to check all possible $w \in \mathbb{Z}/q\mathbb{Z}$
- Clearly infeasible for large q
- Bleichenbacher's solution: reduce the size of h_j's to $[0, L)$ to broaden the peak
  $\rightarrow$ only need to check L evenly-spaced values in $[0, q)$





Small, sparse linear combinations problem

- How do we carry out this range reduction? Linear combinations!
- Input: a list $(h_0, \ldots, h_{L-1})$ of large, random integers (of 160 bits, say)
  - we can choose L, preferably small
- Looking for: many linear combinations $\sum \omega_i h_i$ which are
  - much smaller, e.g. $|\sum \omega_i h_i| < 2^{32}$
  - very sparse, e.g. $\sum |\omega_i| \leq 16$
- We would like to find many of those linear combinations (say 2^{32})
  - as fast as possible
  - using as little memory as possible,
  - starting with as few h_i's as possible

Idea 1: lattice reduction

- "Short linear combinations" sounds like a lattice problem
- So use lattice reduction? (LLL, BKZ)
  - De Mulder et al.'s approach
- Upside: should be able to start from relatively small L
- Downside: only get a few linear combinations for each lattice we reduce + have to use very large lattice dimensions to find the very sparse combinations we need
- Even a single lattice reduction takes seconds with our parameters, and we need $\approx 2^{32}$ of them: not really practical
- Other issue: for ℓ-bit bias, we really need combinations with coefficients $< 2^{\ell}$
  - Doable for $\ell = 5$, infeasible for $\ell = 2$

Idea 2: sort and difference

- Approach in AC'14 paper: sort-and-difference
  1. Sort the list (h_i) to get $h'_0 < \cdots < h'_{L-1}$
  2. Take the successive differences $h''_0 = h'_1 - h'_0, \ldots, h''_{L-2} = h'_{L-1} - h'_{L-2}$
- We obtain a list of $\approx L$ elements h''_i, linear combinations of two elements h_i each
- On average, two successive elements h'_i, h'_{i+1} should have their $\log_2 L$ MSBs in common
- Hence the h''_i are roughly $\log_2 L$ bit shorter
- Doing this 4 times in total yields $\approx L$ linear combinations of 16 elements h_i, each of size $\approx 160 - 4 \times \log_2 L$
- Works with L a bit larger than 2^{32}, in time $O(L \log L)$ and space $O(L)$ (about 1 TB RAM!)

Idea 3: Schroeppel–Shamir

- Shamir's comment after my presentation at ASIACRYPT: you can get away with a smaller L and less memory by using the Schroeppel–Shamir algorithm
- Basic principle: instead of looking for short combinations of 2 h_i at a time with sort-and-difference, we have techniques to generate short(er) combinations of 4 (or more) h_i in one go
- Recently worked this out with an internship student, A. Takahashi
- Surprise realization after we did this: Schroeppel–Shamir was the method suggested by Bleichenbacher all along!

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Schroeppel–Shamir

- At its core, the Schroeppel–Shamir algorithm lets you do the following
- Given two lists (u_i), (v_i) of N integers, find the M smallest sums $u_i + v_j$ ($M \leq N^2$) in time $O((N + M)\log N)$ and space $O(N)$
- Algorithm is not too complicated if we know the heap data structure:
  1. Assume the list (v_i) is sorted (costs $O(N \log N)$ time)
  2. Store the values $u_i + v_0$ associated with the pairs $(i, 0)$ in a heap (costs $O(N \log N)$ time and $O(N)$ space)
  3. Repeat M times:
  4. Get the smallest element in the heap, which is of the form $u_i + v_j$ (associated with (i, j)), and replace it with $u_i + v_{j+1}$ (costs $O(\log N)$ time)

How Schroeppel–Shamir helps

- So remember our original problem: we have a list of integers, and we want to find short linear combinations of four elements at a time (say)
- To do so, divide the large list into 4 lists (x_i), (y_i), (z_i), (t_j) of the same size N
- Schroeppel–Shamir lets us enumerate the elements $x_i + y_j$ and $z_i + t_j$ in increasing order ((L_n) and (R_m) respectively)
- Easy to find short differences between those elements:
  1. Let $m = n = 0$ and $D = L_0 - R_0$
  2. Repeat:
  3. output $D = L_m - R_n$
  4. if $D > 0$ then increment n else increment m
- We thus heuristically get M elements $x_i + y_j - z_k - t_\ell$ which are $\approx \log_2 M$ bit smaller than the original values (or $M/2^s$ elements which are $s + \log_2 M$ bit smaller), in time $O(M \log N)$ and space $O(N)$

How Schroeppel–Shamir helps (II)

- Numerical application: lets say we start with $N = 2^\alpha$ elements of 160 bits, and use Schroeppel–Shamir to get the entire sorted lists of sums, i.e. $M = N^2$
- First iteration: $2^{2\alpha}$ linear combinations of 4 which are of $\lesssim 160 - 2\alpha$ bits, among which we keep the expected 2^α elements of $160 - 3\alpha$ bits or less
- Second iteration: $2^{2\alpha}$ linear combinations of 16 which are of $160 - 5\alpha$ bits or less, among which we keep the expected 2^{32} of $160 - 7\alpha + 32$ bits or less
- We want $160 - 7\alpha + 32 \leq 32$, so $\alpha \gtrsim 23$ should suffice
- We can do all of this in time $\widetilde{O}(2^{46})$ and space $O(2^{23})$: much better than the $O(2^{32})$ memory we started with, at the cost moderate increase in computation
- Possible to minimize the length of the first list even further

Complexity estimate: 160 bits

Bias(bit)	Algorithm	#Round	Time	Space
1	S-S	2	$2^{46.3}$	$2^{25.1}$
	S&D	4	$2^{32.8}$	$2^{32.8}$
2	S-S	3	$2^{38.8}$	$2^{21.4}$
	S&D	6	$2^{23.7}$	$2^{23.7}$
3	S-S	3	$2^{32.8}$	$2^{18.4}$
	S&D	7	$2^{20.9}$	$2^{20.9}$
4	S-S	4	$2^{25.5}$	$2^{14.8}$
	S&D	9	$2^{16.9}$	$2^{16.9}$
5	S-S	5	$2^{21.0}$	$2^{12.5}$
	S&D	11	$2^{14.2}$	$2^{14.2}$

Complexity estimate: 256 bits

Bias(bit)	Algorithm	#Round	Time	Space
1	S-S	2	$2^{73.7}$	$2^{38.9}$
	S&D	5	$2^{43.7}$	$2^{43.7}$
2	S-S	3	$2^{52.0}$	$2^{28.0}$
	S&D	6	$2^{37.4}$	$2^{37.4}$
3	S-S	4	$2^{40.3}$	$2^{22.2}$
	S&D	8	$2^{29.3}$	$2^{29.3}$
4	S-S	5	$2^{38.0}$	$2^{21.0}$
	S&D	10	$2^{24.2}$	$2^{24.2}$
5	S-S	6	$2^{38.0}$	$2^{21.0}$
	S&D	12	$2^{21.0}$	$2^{21.0}$

Work in progress

- Use this approach to mount the 2-bit bias fault attack on qDSA
- Need for large-scale parallelization:
  - not so easy with direct Schroeppel–Shamir (due to heaps)
  - use a simple trick of Howgrave-Graham and Joux to parallelize
  - + some systems design
- More refinements
  - can improve the attack with more data (keep signatures with small h_i's!)
  - can improve the attack with adaptive signature queries (Nikolic–Sasaki, AC'15)
  - asymptotically, can use Generalized Birthday algorithms with more than 4-way collisions

Thank you!

Dank je!